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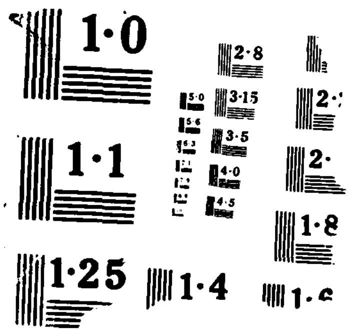
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SYMMETRIES AND BI-HAMILTONIAN STRUCTURES OF 2+1 DIMENSIONAL SYSTEMS

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ABSTRACT

Thetheory associated with the recursion operators of classes of integrable nonlinear evolution equations in 2+1 dimensions is summarized. In particular the notions of symmetries, gradients of conserved quantities, strong and hereditary symmetries, Hamiltonian operators are generalized to equations in multidimensions. Applications to physically relevant equations like the Kadomtsev-Petviashvili equation are illustrated. Integro-differential evolution equations like the Benjamin-Ono equation are shown to be also described by this generalized theory.

A

1. INTRODUCTION

In recent years a great progress in the understanding of the algebraic-geometrical structure of integrable nonlinear evolution equations in 1+1 dimensions has been achieved. A central rôle in the theory is played by an integro-differential operator Φ, the recursion operator. Φ possesses an algebraic-geometrical property, called hereditary [1] or Nijenhuis [2] property, and then it generates commuting symmetries. Moreover: i) its adjoint Φ^* maps gradients of conserved quantities into gradients of conserved quantities; ii) Φ admits a symplectic-cosymplectic factorization and then it generates constants of motion in involution [3]; iii) Φ times the first Hamiltonian $\Theta^{(1)}$ is the second Hamiltonian $\Theta^{(2)}$ [2], then the associated evolution equations have a bi-Hamiltonian nature, a fundamental property underlying integrability [4].

The general theory of recursion operators and their connection to a bi-Hamiltonian formulation in 1+1 dimensions has been developed by Magri [2], Gel'fand and Dorfman [5] and Fokas and Fuchssteiner [3].

An analogous theory for integrable evolution equations in 2+1 dimensions has not been developed prior to our work. In particular no example of multidimensional recursion operator was known (apart from very special cases, like the 2+1 dimensional Burgers equation, that can be linearized through a generalized Cole-Hopf transformation [6]). Various and interesting efforts to obtain recursion operators in 2+1 dimensions were made for instance in [7,8], and essentially showed that recursion operators of a <u>certain form</u> do not exist. It is interesting to notice that a similar situation was known for integro-differential equations in 1+1 dimensions [9], like the Benjamin-Ono equation

$$q_t = Hq_{xx} + 2qq_x, \quad Hf(x) \doteq \pi^{-1} \fint_{-\infty}^{\infty} dx'(x'-x)^{-1} f(x'). \quad (1.1)$$

In this case, in order to bypass the absence of the recursion operator, an alternative approach, the so-called master-symmetries approach, was introduced to generate commuting symmetries [9]. Such an approach has been later successfully used in 2+1 [10] and in 1+1 [12] dimensions, as well as for finite dimensional systems [11].

After the discovery[13] of the recursion operator Φ_{12} and of the two Hamiltonian operators $\Theta^{(1)}_{12}$ and $\Theta^{(2)}_{12}$ associated with the Kadomtsev-Petviashvili (KP) equation

$$q_t = q_{xxx} + 6qq_x + 3\alpha^2 D^{-1} q_{yy}, \tag{1.2}$$

we have developed the theory associated with these operators[14,15] and applied it to two explicit examples: the classes of evolution equations associated with (1.2) and with the Davey-Stewardson equation

$$iq_t + \frac{1}{2}(q_{xx} + \alpha^2 q_{yy}) = q(u - |q|^2),\ u_{xx} - \alpha^2 u_{yy} = 2|q|^2_{xx}. \tag{1.3}$$

In particular, in Ref.14, i) we present a systematic derivation of the recursion operator Φ_{12} from the underlying spectral problem, ii) we show that Φ_{12} generates (what we call) extended symmetries $\sigma^{(m)}_{12}$ and its adjoint generates extended gradients $\gamma^{(m)}_{12}$ of conserved quantities, iii) we show that $\sigma^{(m)}_{12}(q_1,q_2) = 0$ are Bäcklund Transformations (BT) and $\sigma^{(m)}_{11} \doteq \sigma^{(m)}_{12}\Big|_{y_2 = y_1}$ and $\gamma^{(m)}_{11} \doteq \gamma^{(m)}_{12}\Big|_{y_2 = y_1}$ are, respectively, commuting symmetries and gradients of conserved quantities in involution for the associated class of equations. In order to deal with the "extended" objects of this theory, i) we introduce a novel (operator) directional derivative, ii) we deal with a Lie algebra of operators, as opposed to a Lie algebra of functions. In Ref.15, exploiting the richness of this operator Lie algebra, i) we generate, via Φ_{12}, time-dependent symmetries, ii) we use an isomorphism between Lie and Poisson brackets to show that all these symmetries correspond to extended gradients, and hence give rise to conserved quantities, iii) we show that the already known master-symmetries are connected to these symmetries, and, since they correspond to gradients, they cannot be used to generate Φ_{12} ; iv) we find a non-gradient master-symmetry that can be used to generate Φ_{12}, v) we finally motivate and verify some of the aspects of this generalized theory by establishing that equations in 2+1 dimensions are exact reductions of certain nonlocal evolution equations, whose algebraic properties are rather straightforward.

In this note we essentially present a summary of the results contained in [13,14,15], referring to these papers for proofs and details. We additionally exhibit the recursion operator of the Benjamin-Ono (BO) class, showing that integro-differential equations in 1+1 dimensions fit into this generalized theory [16].

2. ALGEBRAIC PROPERTIES IN 2+1 DIMENSIONS

We consider evolution equations of the form

$$q_t = K(q),$$

where q is an element of some space S of functions on the plane vanishing rapidly for $|x|, |y| \to \infty$, and K is some differentiable map on this space (for convenience independent of x,y,t). We use the KP equation (1.2) as an illustrative example.

2.1 Representations Of Integrable Equations In 2+1 Dimensions

Integrable equations in 2+1 dimensions belong to some hierarchy (generated by a recursion operator Φ_{12}); fundamental in our theory is to write these equations in the form

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^n R_{12}^0 \cdot 1 = \beta_n \int_{\mathbb{R}} dy_2 \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}, \qquad (2.1)$$

where $\delta_{12} = \delta(y_1 - y_2)$ denotes the Dirac delta function, $q_i = q(x,y_i,t)$, $i = 1,2$, $K_{12}^{(n)} = K^{(n)}(q_1,q_2) \in \tilde{S}$ and Φ_{12}, R_{12}^0 are operator valued functions on $\tilde{S}$. Through this paper m and n are non-negative integers.

For example the recursion operator associated with the KP class is

$$\Phi_{12} = D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}, \quad q_{12}^{\pm} = q_1 \pm q_2 + \alpha(D_1 \mp D_2), \qquad (2.2)$$

where $D = \partial/\partial_x$, $D^{-1} = \int_{-\infty}^{x} dx'$ and $D_i = \partial/\partial_{y_i}$, $i = 1,2$. This operator generates two classes of evolution equations corresponding to two different starting operators R_{12}^0, given by

$$S_{12} = q_{12}^-, \quad R_{12} = Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-. \qquad (2.3)$$

Then the KP equation is obtained from (2.1) for n = 1, $\beta_1 = 1/2$ and $R_{12}^0 = R_{12}$, while the first two non trivial equations of the second class ($R_{12}^0 = S_{12}$)

are

$$q_t = \alpha q_y, \quad n = 1, \ \beta_1 = 1/4, \tag{2.4a}$$

$$q_t = \alpha(q_{xxy} + 4qq_y + 2q_x D^{-1} q_y + \alpha^2 D^{-2} q_{yyy}), \ n=2, \ \beta_2 = 1/8. \tag{2.4b}$$

The recursion operator Φ_{12} enjoys a simple commutator relation with $h_{12} = h(y_1 - y_2)$:

$$[\Phi_{12}, h_{12}] = -\beta h'_{12}, \quad h'_{12} \doteq \partial h_{12}/\partial y_1, \tag{2.5}$$

which implies that $\delta_{12} K^{(n)}_{12} = \delta_{12} \Phi^n_{12} R^0_{12} \cdot 1 = \sum_{\ell=0}^{n} \beta^\ell \binom{n}{\ell} \Phi^{n-\ell}_{12} \delta_{12} R^0_{12} \cdot 1$, where $\delta^\ell_{12} \doteq \partial^\ell \delta_{12}/\partial y_1^\ell$. Moreover the commutation relation of R^0_{12} with h_{12} implies that $\delta_{12} K^{(n)}_{12}$ can be written in the following <u>alternative form</u>

$$\delta_{12} K^{(n)}_{12} = \sum_{\ell=0}^{n} b_{n,\ell} \Phi^{n-\ell}_{12} R^0_{12} \cdot \delta^\ell_{12} . \tag{2.6}$$

For the two classes of equations associated with the KP equation we have that

$$\begin{gathered} \beta = -4\alpha, \quad [\mathcal{N}_{12}, h_{12}] = 0, \ [\mathcal{M}_{12}, h_{12}] = -\tilde{\beta} D h'_{12}, \ \tilde{\beta} \doteq \beta/2, \\ b_{n,\ell} \doteq \begin{cases} \beta^\ell \binom{n}{\ell}, & \text{if } R^0_{12} = \mathcal{N}_{12}, \\ \sum_{s=0}^{\ell} \beta^{\ell-s} \tilde{\beta}^s \binom{n-s}{\ell-s}, & \text{if } R^0_{12} = \mathcal{M}_{12}. \end{cases} \end{gathered} \tag{2.7}$$

2.2 A New Directional Derivative

In ref.s 14,15 we present a systematic approach to derive recursion operators and classes of integrable 2+1 dimensional equations in the form (2.1) from the underlying spectral problems. This derivation is based on the use of integral representations of operators depending on q and ∂/∂_y. In the KP case, for example, the basic operator

$$\hat{q} = q + \partial/\partial_y \tag{2.8}$$

appearing in the underlying spectral problem

$$w_{xx} + \hat{q} w = 0, \tag{2.9}$$

is represented by

$$\hat{q}_1 f_{12} = (q_1 + \alpha D_1) f_{12} \doteq \int_{\mathbb{R}} dy_3 q_{13} f_{32}, \tag{2.10a}$$

where $g_{ij} \doteq g(x, y_i, y_j, t)$, $i,j = 1,2,3$.

The above mapping between an operator and its kernel induces a mapping between derivatives:

$$\hat{q}_{1_d} [g_{12}] f_{12} = \int_{\mathbb{R}} dy_3 g_{13} f_{32}, \tag{2.10b}$$

where $\hat{q}_{1_d} [g_{12}]$ denotes the directional d-derivative of the operator valued function $\hat{q}_1$ in the direction g_{12}. Using an appropriate bilinear form (see (4.3a)), it is possible to obtain the d-derivative of the adjoint $\hat{q}_1^* = q_2 - \alpha D_2$ of $\hat{q}_1$ and, consequently, of the basic operators $q_{12}^{\pm} = \hat{q}_1 \pm \hat{q}_1^*$:

$$q_{12}^{\pm} [g_{12}] f_{12} = \int_{\mathbb{R}} dy_3 (g_{13} f_{32} \pm f_{13} g_{32}). \tag{2.11}$$

Since Φ_{12} and R_{12}^0 are polynomials in $q_{12}^{\pm}$, their directional derivatives $\Phi_{12_d} [g_{12}]$ and $R^0_{12_d} [g_{12}]$ are well defined.

The d-derivative is a novel aspect of the theory in 2+1 dimensions; its connection with the usual Fréchet derivative (hereafter indicated by the subscript f) is given by the following projective formula

$$K_{12_d} [\delta_{12} g_{12}] = K_{12_f} [g_{12}] \doteq K_{12_{q_1}} [g_{11}] + K_{12_{q_2}} [g_{22}], \tag{2.12}$$

where K_{12} is an arbitrary function in $\tilde{S}$ and $K_{12_{q_i}}$ denotes the Fréchet derivative of K_{12} with respect to q_i, i.e.

$$K_{12_{q_i}} [g_{ii}] = \frac{\partial}{\partial \varepsilon} K_{12}(q_i + \varepsilon g_{ii}, q_j)|_{\varepsilon = 0}, \quad i,j = 1,2;\ i \neq j. \tag{2.13}$$

Operator valued functions on $\tilde{S}$ for which d-derivatives are defined are called admissible.

2.3 The Lie Algebra Of The Starting Symmetries

The starting symmetries K^0_{12} are written as $\mathcal{K}^0_{12}\cdot 1$, where $\mathcal{K}^0_{12}$ are admissible operators. Crucial aspect of this theory is that the operators $\mathcal{K}^0_{12}$, acting on suitable functions H_{12} belonging to the Ker of the first Hamiltonian operator $\Theta^{(1)}_{12}$ (i.e. such that $\Theta^{(1)}_{12}H_{12}=0$), form a Lie algebra. Then, with respect to the 1+1 dimensional case characterized by a Lie algebra of functions, equations in 2+1 dimensions present a richer algebraic structure, characterized by a <u>Lie algebra of operators.</u>

For the equations associated with the KP equation, H_{12} is an arbitrary function independent of x, i.e.

$$H_{12}= H(y_1,y_2), \tag{2.14}$$

and the Lie algebra of the starting operators $\mathcal{S}_{12}$ and $\mathcal{R}_{12}$ is given by

$$[\mathcal{S}_{12}H^{(1)}_{12}, \mathcal{S}_{12}H^{(2)}_{12}]_d = -\mathcal{S}_{12}H^{(3)}_{12}, \quad [\mathcal{S}_{12}H^{(1)}_{12}, \mathcal{R}_{12}H^{(2)}_{12}]_d = -\mathcal{R}_{12}H^{(3)}_{12},$$
$$[\mathcal{R}_{12}H^{(1)}_{12}, \mathcal{R}_{12}H^{(2)}_{12}]_d = -\Phi_{12}\mathcal{S}_{12}H^{(3)}_{12}, \tag{2.15}$$

in terms of the Lie brackets $[\cdot]_d$, $[\ ,\ ]_I$, defined by

$$[K^{(1)}_{12}, K^{(2)}_{12}]_d \doteq K^{(1)}_{12_d}[K^{(2)}_{12}] - K^{(2)}_{12_d}[K^{(1)}_{12}], \tag{2.16a}$$

$$H^{(3)}_{12} \doteq [H^{(1)}_{12}, H^{(2)}_{12}]_I \doteq \int_{\mathbb{R}} dy_3(H^{(1)}_{13}H^{(2)}_{32} - H^{(2)}_{13}H^{(1)}_{32}). \tag{2.16b}$$

3. SYMMETRIES AND HEREDITARY SYMMETRIES

3.1 The Notion Of Extended Symmetries, Connection To Symmetries And Bäcklund Transformations

The recursion operator Φ_{12} generates a sequence of functions $\sigma^{(m)}_{12}= \sigma^{(m)}(q_1,q_2) \in \tilde{S}$ defined by

$$\sigma^{(m)}_{12} \doteq \Phi^m_{12}\,\mathcal{K}^0_{12}\cdot 1. \tag{3.1}$$

These functions live in the extended space $\tilde{S}$ and, in order to give them a characterization and establish their connection to the integrable evolution equation

$$q_{1_t} = \int_{\mathbb{R}} dy_2\delta_{12}K_{12}= K_{11}, \tag{3.2}$$

the following definition is introduced.

Definition 3.1

The function $\sigma_{12} \in \tilde{S}$ is called extended symmetry of equation (3.2) iff

$$\sigma_{12\,f}[K] = (\delta_{12} K_{12})_d [\sigma_{12}]. \tag{3.3}$$

Remark 3.1

i) Definition (3.3) makes sense only if $(\delta_{12} K_{12})_d$ exists; for equations $q_{1_t} = \int_{\mathbb{R}} dy_2\, \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}$ generated by Φ_{12}, $(\delta_{12} K_{12})_d$ is well defined and reads

$$(\delta_{12} K_{12}^{(n)})_d = (\delta_{12} \Phi_{12}^n\ R_{12}^{\circ} \cdot 1)_d = \sum_{\ell=0}^{n} b_{n\ell}\ (\Phi_{12}^{n-\ell} R_{12}^{\circ} \cdot \delta_{12}^{\ell})_d, \tag{3.4}$$

where equation (2.6) is clearly used to write $\delta_{12} K_{12}^{(n)}$ in admissible form. If, for example, $R_{12}^{\circ} = R_{12}$ and n=0 and 1, we have

$$(\delta_{12} K_{12}^{(0)})_d = 2D, \tag{3.5a}$$

$$(\delta_{12} K_{12}^{(1)})_d = 2(D^3 + 6D(q_1+q_2) - 3\alpha(D^{-1}(q_{1_{y_1}} - q_{2_{y_2}})) + 6\alpha(q_1-q_2)D^{-1}(D_1+D_2) + 6\alpha^2 D^{-1}(D_1+D_2)^2). \tag{3.5b}$$

ii) The projective property (2.12) implies that σ_{12} is an extended symmetry of (3.2) iff σ_{12} commutes with $\delta_{12} K_{12}$, namely iff

$$[\sigma_{12},\ \delta_{12} K_{12}]_d = 0. \tag{3.6}$$

iii) In the above definition we assume that σ_{12} does not explicitely depend on t, otherwise $\sigma_{12\,f}[K]$ should be replaced by $\partial\sigma_{12}/\partial t + \sigma_{12\,f}[K]$.

The usefulness of extended symmetries follows from the fact that they give rise to symmetries and Bäklund transformations. Precisely we have the following

Theorem 3.1

If σ_{12} is an extended symmetry of equation (3.2), then

i) $\sigma_{11} = \sigma_{12}|_{y_2=y_1}$ is a symmetry of equation (3.2), namely

$$\sigma_{11_f}[K_{11}] = K_{11_f}[\sigma_{11}]; \tag{3.7a}$$

ii) the equation

$$\sigma_{12} = \sigma(q_1, q_2) = 0 \tag{3.7b}$$

is a Bäcklund transformation for (3.2) where, of course, q_1 and q_2 are

are now viewed as two different solutions of (3.2).

3.2 Strong And Hereditary Symmetries

The introduction of the d-derivative allows a natural generalization of the notions of strong and hereditary symmetries in 2+1 dimensions; precisely we have the following

Definition 3.2

i) The admissible operator valued function Φ_{12} is a strong symmetry for A_{12} iff

$$\Phi_{12_d}[A_{12}] + [\Phi_{12}, A_{12_d}] = 0. \tag{3.8}$$

ii) The admissible operator valued function Φ_{12} is called hereditary symmetry iff

$$\Phi_{12_d}[\Phi_{12} f_{12}]\, g_{12} - \Phi_{12}\, \Phi_{12_d}[f_{12}]\, g_{12} \text{ is symmetric w.r.t. } f_{12}, g_{12}. \tag{3.9}$$

Remark 3.2

i) Equation (3.8) makes sense if the d-derivative of A_{12} exists; if A_{12} is generated by an admissible operator $\hat{A}_{12}$ on H_{12} ($A_{12} = \hat{A}_{12} H_{12}$), then A_{12} is admissible and $A_{12_d}[\cdot] = \hat{A}_{12_d}[\cdot] H_{12}$. Since a linear combination of admissible functions is admissible, the equation (2.6) implies that $\delta_{12} K_{12}^{(n)}$ is an admissible function.

ii) If $A_{12} = \delta_{12} K_{12}$, then equation (3.8) reads

$$\Phi_{12_f}[K] + [\Phi_{12}, (\delta_{12} K_{12})_d] = 0, \tag{3.10}$$

and we say that Φ_{12} is a strong symmetry of the evolution equation (3.2). In this case Φ_{12} maps extended symmetries of (3.2) to extended symmetries of (3.2).

As for equations in 1+1, hereditary operators generate infinitely many commuting symmetries; we have precisely the following

Theorem 3.2

If the hereditary operator Φ_{12} is a strong symmetry of the starting symmetry $R_{12}^{o} H_{12}$, and if the starting operator R_{12}^{o} satisfies the following condition

$$\left[R^{o}_{12}H^{(1)}_{12},\ R^{o}_{12}H^{(2)}_{12} \right]_{d} = 0 \qquad \text{for} \qquad \left[H^{(1)}_{12},\ H^{(2)}_{12}\right]_{I} = 0, \tag{3.11}$$

then $\sigma^{(m)}_{12} \doteq \Phi^{m}_{12} R^{o}_{12} \cdot 1$ are extended symmetries of every evolution equation (2.1).

Corollary 3.1

If Φ_{12} generates two classes of evolution equations corresponding to two starting operators R^{o}_{12}, given by $\mathfrak{N}_{12}$ and $\mathfrak{A}_{12}$, and if $\mathfrak{N}_{12}$ and $\mathfrak{A}_{12}$ satisfy a Lie algebra of the type (2.15,16), then $\Phi^{m}_{12} \mathfrak{N}_{12} \cdot 1$ and $\Phi^{m}_{12} \mathfrak{A}_{12} \cdot 1$ are extended symmetries of both classes of evolution equations (2.1).

It turns out that the recursion operator (2.2) is hereditary and is a strong symmetry of the starting symmetries (2.3), then it follows that $\sigma^{(m)}_{12} = \Phi^{m}_{12} R^{o}_{12} \cdot 1$ (for $R^{o}_{12} = \mathfrak{N}_{12}$ and/or $\mathfrak{A}_{12}$) are extended symmetries of each equation of the KP classes, and Theorem 3.1 implies that $\sigma^{(m)}_{11}$ are symmetries and $\sigma^{(m)}_{12} = \sigma^{(m)}(q_1,q_2) = 0$ are BT of each member of the two hierarchies.

3.3 Isospectral Problems Yield Hereditary Symmetries

The previous section illustrates the importance of hereditary symmetries. For equations in 1+1 and in 2+1 dimensions isospectral problems yield hereditary operators:

Proposition 3.1

Let

$$\frac{dV}{dx} = U(\hat{q},\lambda)V \tag{3.12}$$

be an isospectral two dimensional problem, where $\hat{q}$ is an operator depending on $q(x,y)$ and $\partial/\partial y$, and λ is an eigenvalue. Assume that $(G_{\lambda})_{12}$, the extended gradient of λ (see (4.2)), satisfies

$$\Psi_{12}(G_{\lambda})_{12} = \mu(\lambda)(G_{\lambda})_{12}. \tag{3.13}$$

Then if $\Phi_{12} = \Psi^{*}_{12}$ has a complete set of eigenfunctions, it is hereditary.

For example the isospectral problem

$$v_{xx}+\tilde{q}v = \lambda v \tag{3.14}$$

associated with the KP classes implies that

$$(G_\lambda)_{12} = v_1 v_2^+, \tag{3.15}$$

where v^+ is a solution of $v_{xx}^+ + (q - \alpha\partial/\partial_y)v^+ = \lambda v^+$. Since Φ_{12}, defined by (2.2), satisfies

$$\Phi_{12}^* v_1 v_2^+ = 4\lambda v_1 v_2^+, \tag{3.16}$$

it follows that Φ_{12} is hereditary.

4. HAMILTONIAN FORMALISM

4.1 Bilinear Forms, Gradients Of Conserved Quantities

Integrable evolution equations in 2+1 dimensions posses infinitely many constants of motion of the form

$$I = \mathrm{tr} \int_{\mathbb{R}^2} dxdy_1 \rho_{11} = \mathrm{tr} \int_{\mathbb{R}^3} dxdy_1 dy_2 \delta_{12} \rho_{12} , \tag{4.1}$$

where $\rho_{12} = \rho(q_1, q_2)$ (the trace operation is obviously omitted if q is a scalar).

As in 1+1 dimensions, it is more convenient to deal with the gradients of conserved quantities; in this case the double representation (4.1) of a functional I allows the introduction of the extended gradient $\mathrm{grad}_{12} I$ and of the gradient grad I of I, defined by

$$I_d[g_{12}] = \mathrm{tr} \int_{\mathbb{R}^3} dxdy_1 dy_2 \delta_{12} \rho_{12_d}[g_{12}] \doteq < \mathrm{grad}_{12} I, g_{12} > , \tag{4.2a}$$

$$I_f[g_{11}] = \mathrm{tr} \int_{\mathbb{R}^2} dxdy_1 \rho_{11_f}[g_{11}] \doteq (\mathrm{grad}\ I, g_{11}), \tag{4.2b}$$

where

$$< g_{12}, f_{12} > \doteq \mathrm{tr} \int_{\mathbb{R}^3} dxdy_1 dy_2 g_{21} f_{12}, \tag{4.3a}$$

$$(g_{11}, f_{11}) \doteq \mathrm{tr} \int_{\mathbb{R}^2} dxdy_1 g_{11} f_{11}, \tag{4.3b}$$

are the proper symmetric bilinear forms coupling arbitrary elements $g_{12} \in \tilde{S}^*$, $f_{12} \in \tilde{S}$ and $g_{11} \in S^*$, $f_{11} \in S$ respectively ($\tilde{S}^*$ and S^* are obviously the duals of $\tilde{S}$ and S).

If L_{12}^* and L^+ denote the adjoints of the operator valued functions L_{12} and L on $\tilde{S}$ and S with respect to the bilinear forms (4.3a) and (4.3b) respectively, namely if

$$< L^{*}_{12} g_{12},\ f_{12} > \doteq \ <g_{12},\ L_{12} f_{12}> , \tag{4.4a}$$

$$(L^{+} g_{11},\ f_{11}) \doteq \ (g_{11},\ L f_{11}), \tag{4.4b}$$

then one can prove the following

Proposition 4.1

i) γ_{12} and γ_{11} are extended gradient and gradient functions respectively, iff $\gamma^{*}_{12d} = \gamma_{12d}$ and $\gamma^{+}_{11f} = \gamma_{11f}$.

ii) If γ_{12} is an extended gradient, then γ_{11} is a gradient corresponding to the same potential, namely if $\gamma_{12} = \mathrm{grad}_{12} I$, then $\gamma_{11} = \mathrm{grad}\ I$.

If a functional I is conserved with respect to the evolution equation (3.2), this corresponds to the mathematical notion of conserved covariant.

Definition 4.1

The function γ_{12} is an extended conserved covariant of (3.2) iff

$$\gamma_{12_f}[K] + (\delta_{12} K_{12})^{*}_{d}[\gamma_{12}] = 0. \tag{4.5}$$

Then we have the following

Proposition 4.2

i) If γ_{12} is an extended conserved covariant of (3.2), γ_{11} is a conserved covariant of (3.2), namely

$$\gamma_{11_f}[K_{11}] + K^{*}_{11_f}[\gamma_{11}] = 0. \tag{4.6}$$

ii) If the functional I is a conserved quantity of (3.2), then $\gamma_{12} = \mathrm{grad}_{12} I$ is an extended conserved covariant of (3.2). Conversely, if γ_{12} is an extended conserved covariant of (3.2) and it is the extended gradient of a functional I, then I is a constant of motion for (3.2).

As an illustration of the notions presented in this section, we have that the functionals I_j defined by

$$I_j = \frac{1}{2(2j+3)} \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12}\ \gamma_{12}^{(j+1)}\ ,\ j=0,1, \tag{4.7}$$

where

$$\gamma_{12}^{(j)} = D^{-1} \Phi_{12}^{\ j}\ R^{o}_{12} \cdot 1\ ,\ R^{o}_{12} = R_{12} \text{ and } R_{12}. \tag{4.8}$$

are constants of motion of the KP hierarchies, corresponding tc the extended gradients of conserved quantities

$$\gamma_{12}^{(j)} = \mathrm{grad}_{12} I_j = D^{-1} \Phi_{12}^{j} R_{12}^{\circ} \cdot 1 \ , \quad j=o,1. \tag{4.9}$$

Moreover $q_{12}^{\pm *} = \pm q_{12}^{\pm}$, then

$$\Phi_{12}^{*} = D^2 + D^{-1} q_{12}^{+} D + q_{12}^{+} + D^{-1} q_{12}^{-} D^{-1} q_{12}^{-}. \tag{4.10}$$

4.2 Bi-Hamiltonian Structures

The existence of a recursion operator for 2+1 dimensional systems allows a characterization of their bi-Hamiltonian nature. The "extended nature" of the operators we are dealing with, leads to a definition of Hamiltonian system in an "extended sense".

Definition 4.2

An equation (3.2) is of a Hamiltonian form (or is a Hamiltonian system) if it can be written as

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \ \delta_{12} \Theta_{12} \gamma_{12} \ , \tag{4.11}$$

where γ_{12} is an extended gradient function of the form $\gamma_{12} = \hat{\gamma}_{12} \cdot 1$ and Θ_{12} is a Hamiltonian operator, i.e.

i) $\Theta_{12}^{*} = - \Theta_{12}$, (4.12a)

ii) Θ_{12} satisfies the Jacobi identity w.r.t. the bracket

$$\{a_{12}, b_{12}, c_{12}\} \doteq < a_{12}, \ \Theta_{12} \ d\left[\Theta_{12} b_{12}\right] c_{12} >^{\sim}. \tag{4.12b}$$

The associated Poisson bracket of two functionals $I^{(1)}$ and $I^{(2)}$ is given by

$$\{ I^{(1)}, I^{(2)} \} \doteq < \mathrm{grad}_{12} I^{(1)}, \Theta_{12} \ \mathrm{grad}_{12} I^{(2)} > . \tag{4.13}$$

$\Theta_{12}^{(1)} = D$ and $\Theta_{12}^{(2)} = \Phi_{12} D$, where Φ_{12} is defined in (2.2), are examples of Hamiltonian operators. Then, since $\gamma_{12}^{(0)} = D^{-1} R_{12} \cdot 1$ and $\gamma_{12}^{(1)} = D^{-1} \Phi_{12} R_{12} \cdot 1$ are extended gradients, the KP equation has a double Hamiltonian structure (is a bi-Hamiltonian system):

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \ \Theta_{12}^{(1)} \ \gamma_{12}^{(1)} = \int_{\mathbb{R}} dy_2 \delta_{12} \ \Theta_{12}^{(2)} \gamma_{12}^{(0)} \ . \tag{4.14}$$

As in 1+1 dimensions, the existence of a compatible pair of Hamiltonian operators plays a fundamental rôle in the theory, emphasized by the following Theorem

Theorem 4.1

Let $\Theta^{(1)}_{12} + \nu\Theta^{(2)}_{12}$ be a Hamiltonian operator for all constant values of ν. Assume that $\Theta^{(1)}_{12}$ is invertible. Define $\Phi_{12} \doteq \Theta^{(2)}_{12}(\Theta^{(1)}_{12})^{-1}$, $\hat{\gamma}^{o}_{12} \doteq (\Theta^{(1)}_{12})^{-1}\hat{K}^{o}_{12}$. Assume that the operator Φ_{12} is a strong symmetry for the starting symmetry $\hat{K}^{o}_{12}H_{12}$ that satisfies (3.11); further assume that $\hat{\gamma}^{o}_{12} H_{12}$ is an extended gradient function. Then

i) Equations (2.1) are bi-Hamiltonian systems.

ii) Φ is hereditary.

iii) $K^{(m)}_{12} \doteq \Phi^{m}_{12}\hat{K}_{12}\cdot 1$ and $\gamma^{(m)}_{12} \doteq (\Phi^{*}_{12})^{m}\,\hat{\gamma}^{o}_{12}\cdot 1$ are extended symmetries and extended gradients of conserved quantities respectively of equations (2.1).

iv) $K^{(m)}_{11}$ and $\gamma^{(m)}_{11}$ are symmetries and gradients of conserved quantities in involution for equations (2.1), namely

$$K^{(m)}_{11_f}\left[K^{(n)}_{11}\right] = K^{(n)}_{11_f}\left[K^{(m)}_{11}\right], \tag{4.15a}$$

$$\{I^{(m)}, I^{(n)}\} = \langle\delta_{12}\,\gamma^{(m)}_{12}, \Theta_{12}\,\gamma^{(n)}_{12}\rangle = 0,\ \Theta_{12} = \Theta^{(1)}_{12} \text{ or } \Theta^{(2)}_{12}, \tag{4.15b}$$

$$\gamma^{(j)}_{12} \doteq \mathrm{grad}_{12} I^{(j)}. \tag{4.15c}$$

v) $K^{(m)}_{12} \doteq K^{(m)}(q_1,q_2) = 0$ are auto-BT for equations (2.1).

Remark 4.1

If $\Phi_{12} = \Theta^{(2)}_{12}(\Theta^{(1)}_{12})^{-1}$, where $\Theta^{(1)}_{12}$ and $\Theta^{(2)}_{12}$ are skew-symmetric, then $\Phi_{12}\Theta^{(1)}_{12} = \Theta^{(1)}_{12}\Phi^{*}_{12}$ (the operators Φ_{12} and $\Theta^{(1)}_{12}$ are well coupled).

The hypothesis of Theorem 4.1 are satisfied by the Hamiltonian operators $\Theta^{(1)}_{12}$, $\Theta^{(2)}_{12}$ and by the starting operators $\hat{K}_{12}$ and $\hat{A}_{12}$ of the KP classes; then they enjoy the propetries i)-v).

5. OTHER ASPECTS OF THE THEORY

5.1 Time Dependent Symmetries, Connection To Master-Symmetries

In order to investigate the properties of time independent commuting symmetries of the integrable evolution equations (2.1), one uses only special choices of H_{12} (given by $H_{12} = 1$ and δ^{ℓ}_{12}) for which the Lie algebra of the starting symmetries is abelian. More general choices of H_{12} make the Lie algebra of the starting symmetries non-abelian and give rise to time dependent extended symmetries.

A time dependent symmetry

$$\sigma_{12} = \sum_{j=0}^{r} t^j \, \Sigma^{(j)}_{12} \tag{5.1}$$

of equation (3.2) must satisfy the following equations

$$\Sigma^{(j)}_{12} = -\frac{1}{j}\left[\Sigma^{(j-1)}_{12}, \delta_{12} K_{12}\right]_d, \quad j=1,\ldots,r, \tag{5.2a}$$

$$\left[\Sigma^{(r)}_{12}, \delta_{12} K_{12}\right]_d = 0. \tag{5.2b}$$

This implies that constructing a symmetry of order r in time is equivalent to finding a function $\Sigma^{(0)}_{12}$ with the property that its $r+1^{st}$ Lie commutator with $\delta_{12} K_{12}$ is zero. Considering the KP class and using the structure of the Lie algebra (2.15,16) and equation (2.6), it is possible to show that t-dependent symmetries of order r are generated through equations (5.2) starting with

$$\Sigma^{(0)}_{12} = \tau^{(m,r)}_{12} \doteq \Phi^{m}_{12} R^{o}_{12} H^{(r)}_{12}, \tag{5.3}$$

where $R^{o}_{12} = S_{12}$ and/or R_{12}, and $H^{(r)}_{12}$ is defined by

$$H^{(r)}_{12} \doteq (y_1+y_2)^r, \tag{5.4}$$

or, more generally, by any homogeneous polynomial of degree r in y_1 and y_2. For example σ_{12} given by (5.1), with

$$\Sigma^{(0)}_{12} = \Phi^{m}_{12} R^{o}_{12} H^{(r)}_{12}, \tag{5.5a}$$

$$\Sigma^{(j)}_{12} \doteq \sum_{s_1,\ldots,s_j=0}^{p_n} \nu(r,j,s)\, \Phi_{12}^{(m+jn-\sum_{\ell=1}^{j}(2s_\ell+1))} R_{12} H_{12}^{(r-\sum_{\ell=1}^{j}(2s_\ell+1))}, \tag{5.5b}$$

$$\nu(r,j,s) \doteq \frac{(-2)^j}{j!}\left(\prod_{\mu=1}^{j} \theta\left(r-\sum_{\ell=1}^{\mu}(2s_\ell+1)\right)\right)\left(\sum_{\ell=1}^{j} b_{n,2s_\ell+1}\right)\frac{r!}{\left(r-\sum_{\ell=1}^{j}(2s_\ell+1)\right)!}, \tag{5.5c}$$

$$\theta(a) \doteq \begin{cases} 1, & a \geq 0 \\ 0, & a < 0 \end{cases}, \quad p_n \doteq \begin{cases} (n-1)/2, & n \text{ odd} \\ (n-2)/2, & n \text{ even} \end{cases} \quad b_{n,\ell} \doteq (-4\alpha)^{\ell}\binom{n}{\ell}, \tag{5.5d}$$

is a time dependent extended symmetry of order r of equation $(2.1)_n$, corresponding to $K_{12}^{0} = S_{12}$. The discovery of the existence of a hereditary operator Φ_{12}, together with the structure of the Lie algebra of the starting symmetries, allows a simple and elegant characterization of the 2+1 dimensional master-symmetries. Here we briefly remark that

$$\tau_{11}^{(m,r)} \doteq \int_{\mathbb{R}} dy_2\, \delta_{12}\, \tau_{12}^{(m,r)} \tag{5.6}$$

are the so-called master-symmetries of degree r of KP [12].

5.2 Gradient And Non-Gradient Master-Symmetries

Using an isomorphism between Lie and Poisson brackets and the Lie algebra of the starting symmetries $K_{12}^{o}H_{12}$, it is easy to prove that $(\Theta_{12}^{(1)})^{-1}\Phi_{12}^{m} K_{12}^{o}H_{12}$ are extended gradients. This implies that the extended symmetries associated with them give rise to conserved quantities. For example, the t-dependent symmetry

$$\sigma_{12} = \mathfrak{A}_{12}^{(m)}(y_1+y_2) + t12\alpha\, \mathfrak{S}_{12}^{(m+1)}\cdot 1 \tag{5.7}$$

of the KP equation $q_t = 2(q_{xxx} + 6qq_x + 3\alpha^2 D^{-1} q_{yy})$ corresponds to the extended gradient $D^{-1}\sigma_{12}$, then it gives rise to the t-dependent conserved quantity

$$I = \int_{\mathbb{R}^2} dx dy_1 \Big(\frac{1}{2(2m+3)} (D^{-1}\mathfrak{A}_{12}^{(m+1)}(y_1+y_2))_{y_2=y_1} + \frac{3\alpha t}{m+2} (D^{-1}\mathfrak{S}_{12}^{(m+2)}\cdot 1)_{y_2=y_1} \Big). \tag{5.8}$$

Since the master-symmetries $\tau_{12}^{(m,r)}$ are related to these gradients, they cannot be used to generate Φ_{12}. Nevertheless non-gradient master-symmetries of 2+1 dimensional equations exist, for example $T_{12} = \Phi_{12}^{2} S_{12}$ is a non-gradient master-symmetry of the KP classes, satisfying the following equation

$$[\Phi_{12}^{n} K_{12}^{o}\cdot 1,\, T_{12}]_d = b_n\, \Phi_{12}^{n+1} K_{12}^{o}\cdot 1, \tag{5.9}$$

where $b_n = 4n$ and $2(2n+1)$, if $R_{12}^{?} = \mathfrak{I}_{12}$ and $\mathfrak{R}_{12}$ respectively.

Since $(\Theta_{12}^{(1)})^{-1} T_{12}$ is <u>not</u> a gradient function, T_{12} can be used to generate Φ_{12}, exactly as in 1+1 dimensions, through the formula

$$8\Phi_{12} = T_{12_d} + \Theta_{12}^{(1)} T_{12_d} (\Theta_{12}^{(1)})^{-1}. \tag{5.10}$$

The existence of the T_{12} master-symmetry finally implies a simple derivation of the equation

$$I_n = \frac{1}{b_{n+1}} \int_{\mathbb{R}^2} dx dy_1 \gamma_{11}^{(n+1)}, \tag{5.11}$$

where I_n is the potential of $\gamma_{11}^{(n)}$ ($\gamma_{11}^{(n)} = \text{grad } I_n$), and $\gamma_{12}^{(j)} = (\Phi_{12}^{*})^{j} \Theta_{12}^{-1} R_{12}^{0} \cdot 1$.

6. THE BENJAMIN-ONO CLASS

The remarkable connections (algebraic and analytical) between the KP and the BO equations are also confirmed by the fact that, although the BO equation is a 1+1 dimensional system, its recursion operator lives in an extended space. In fact it is possible to show that the BO class can be represented in the following way[6]

$$q_{1_t} = B_n \int_{\mathbb{R}} dx_2 \delta_{12} \Phi_{12}^{n} q_{12}^{-} \cdot 1, \tag{6.1}$$

where $\delta_{12} = \delta(x_1 - x_2)$. The hereditary operator Φ_{12} is defined by

$$\Phi_{12} = q_{12}^{+} - i q_{12}^{-} H, \quad q_{12}^{\pm} = q_1 \pm q_2 + i(D_1 \mp D_2),$$
$$q_i = q(x_i, t), \; D_i = \partial/\partial x_i, \; i = 1,2, \tag{6.2}$$

and the operator H, whose action is defined on functions $f_{12} = f(x_1, x_2)$ of the type

$$f_{12} = a_1 + b_2 + c_{12}, \; H_1 c_{12} = H_2 c_{12}, \tag{6.3}$$

is given by

$$H f_{12} = H_1 a_1 + H_2 b_2 + H_1 c_{12}, \tag{6.4}$$

where H_i, $i=1,2$, is the Hilbert transform with respect to the variable

x_i :

$$H_i g_{ij} = \pi^{-1} \int_R dx_i' (x_i' - x_i)^{-1} g(x_i', x_j), \qquad i \neq j . \tag{6.5}$$

It is possible to show that all the extended symmetries $\sigma_{12}^{(j)}$ generated by Φ_{12} :

$$\sigma_{12}^{(j)} \doteq \Phi_{12}^{\ j} q_{12}^{-} \cdot 1 \tag{6.6}$$

are functions of the type (6.3), then formula (6.1) is well defined. The BO equation (1.1) corresponds to $n=2$ and $\beta_2=(4i)^{-1}$. One can show that the algebraic properties of the BO class (6.1) are described by the generalized theory summarized in this paper.

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